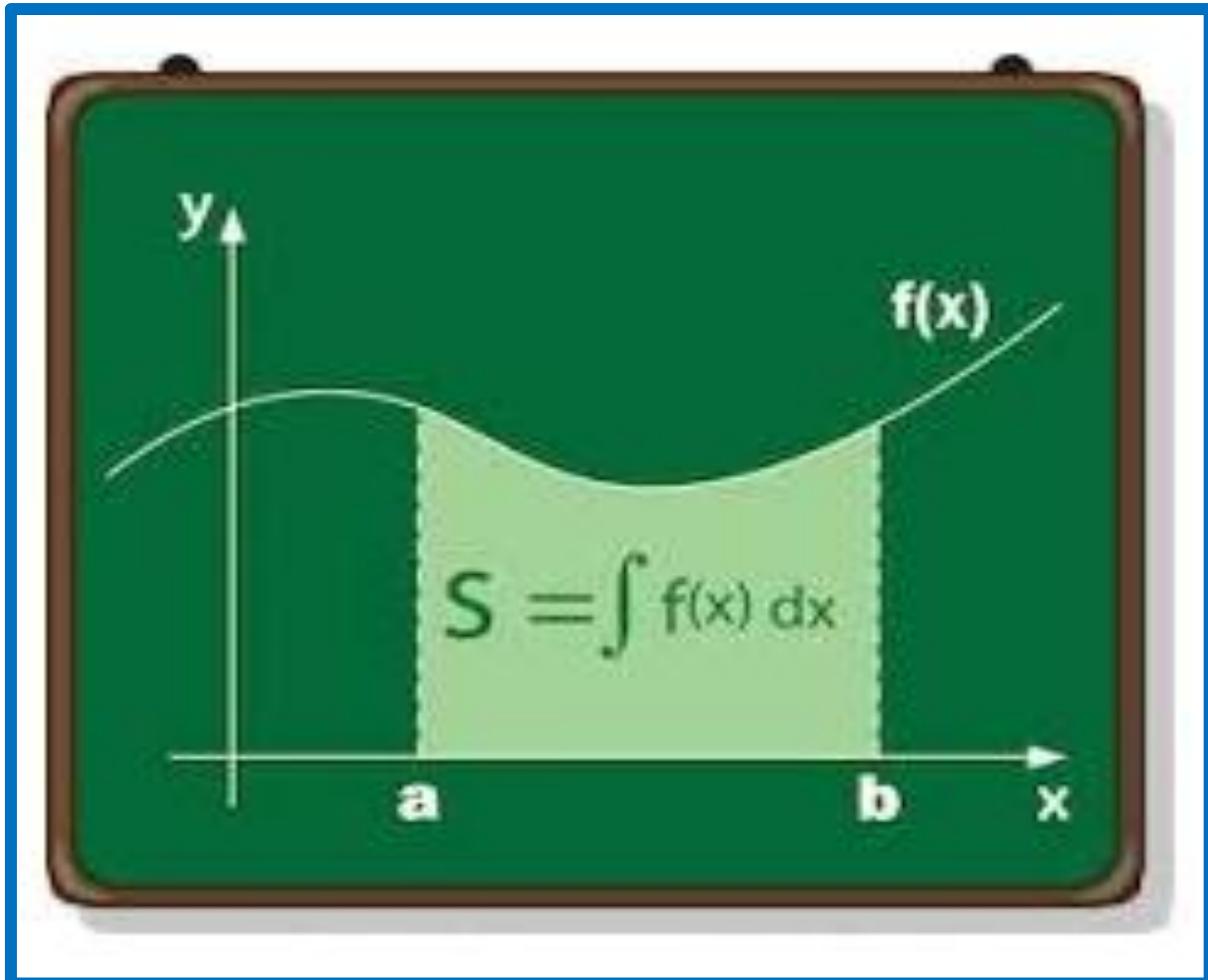
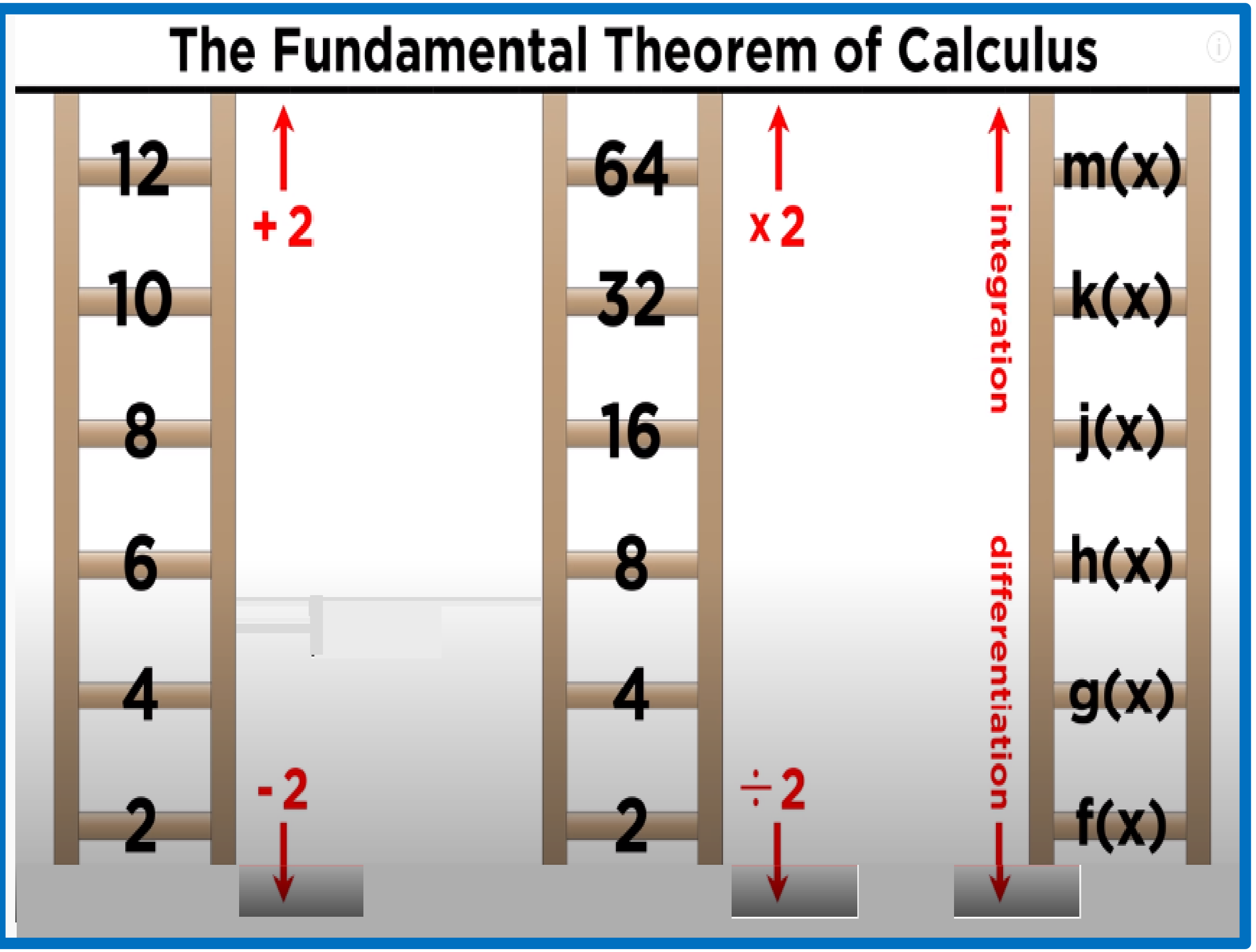




INTEGRALS

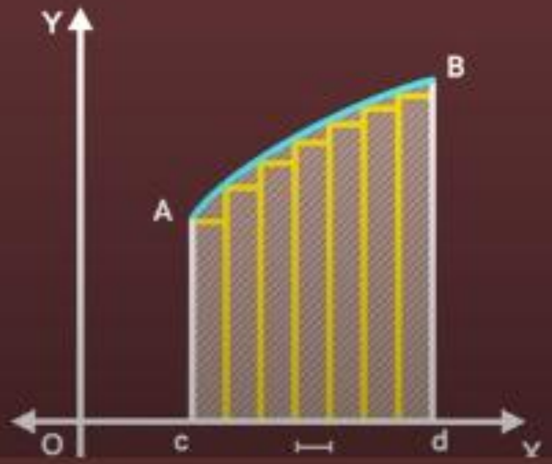
MODULE – 8

DIFFERENTIATION ⇆ INTEGRATION

Derivative of a function ▶ Instantaneous rate of change

Integral of a function ▶ Area under its graph

INTEGRAL OF A FUNCTION



Sum (areas of the rectangles) $\xrightarrow{\Delta x \to 0}$ Area under the curve

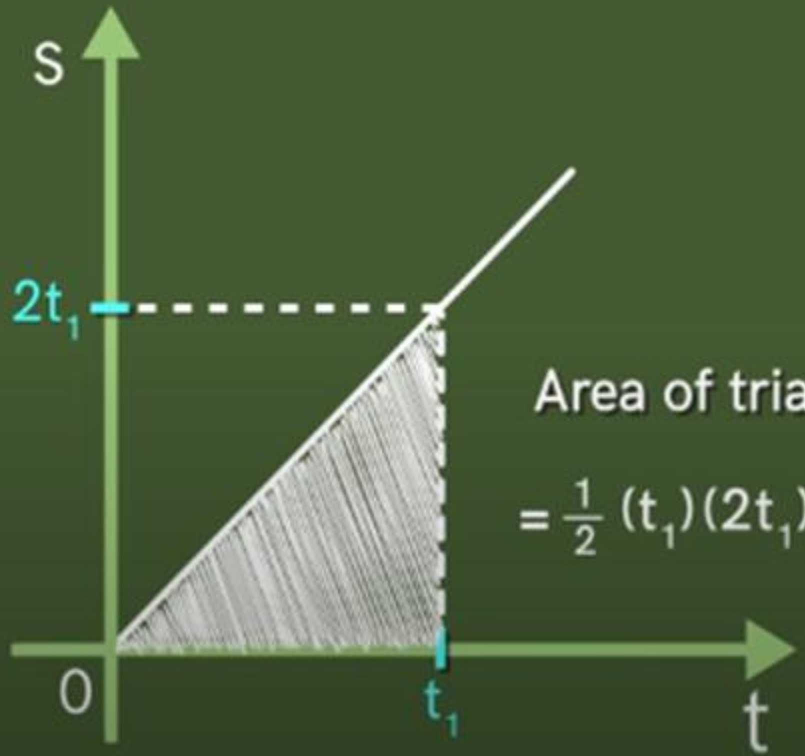
DISTANCE FUNCTION : $y=f(t)=t^2$

SPEED FUNCTION : $s=f'(t)=2t$

y : distance travelled in metres(m)

s : speed (m/s)

t : time in seconds (s)



Area of triangle

$$= \frac{1}{2}(t_1)(2t_1) = t_1^2$$

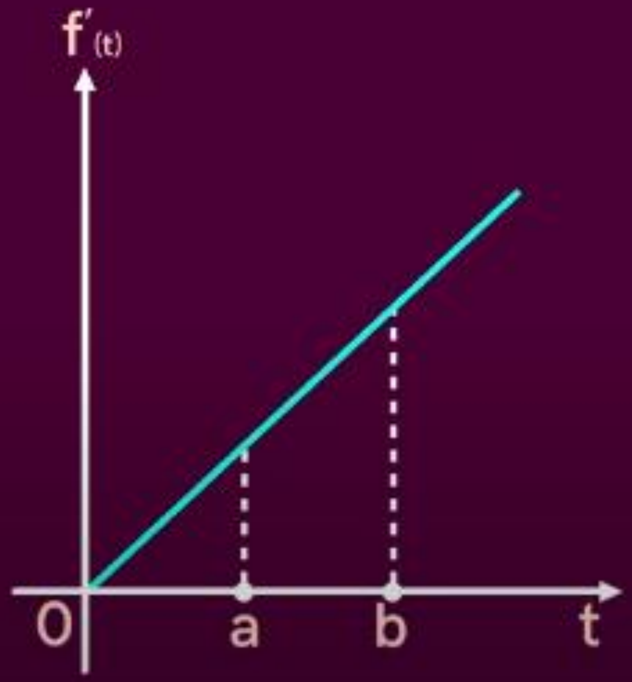
$$\int_0^{t_1}(2t)dt = t_1^2 = F(t_1)$$

Distance travelled in time t_1

Area under graph → Distance travelled of Speed function

$$f(t) = t^2 \xrightarrow{\text{Differentiation}} f'(t) = 2t$$

$$f(t_1) = (t_1)^2 = \int_0^{t_1} (2t)\,dt \xleftarrow{\text{Integration}} f'(t) = 2t$$



$$\int_a^b (2t)\,dt = \int_0^b (2t)\,dt - \int_0^a (2t)\,dt$$

$$= f(b) - f(a)$$

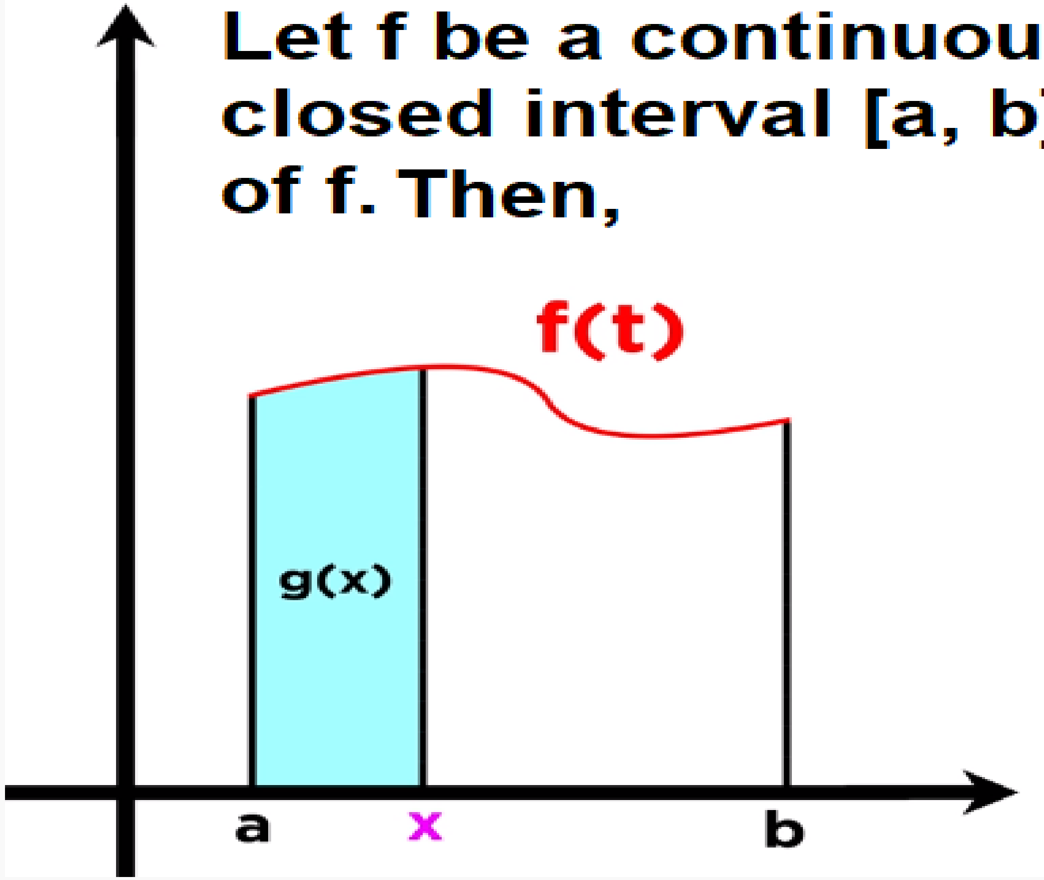
$$= b^2 - a^2$$

Fundamental theorem of Calculus

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

The Fundamental Theorem of Calculus

Let f be a continuous function defined on the closed interval [a, b] and F be an anti-derivative of f. Then,



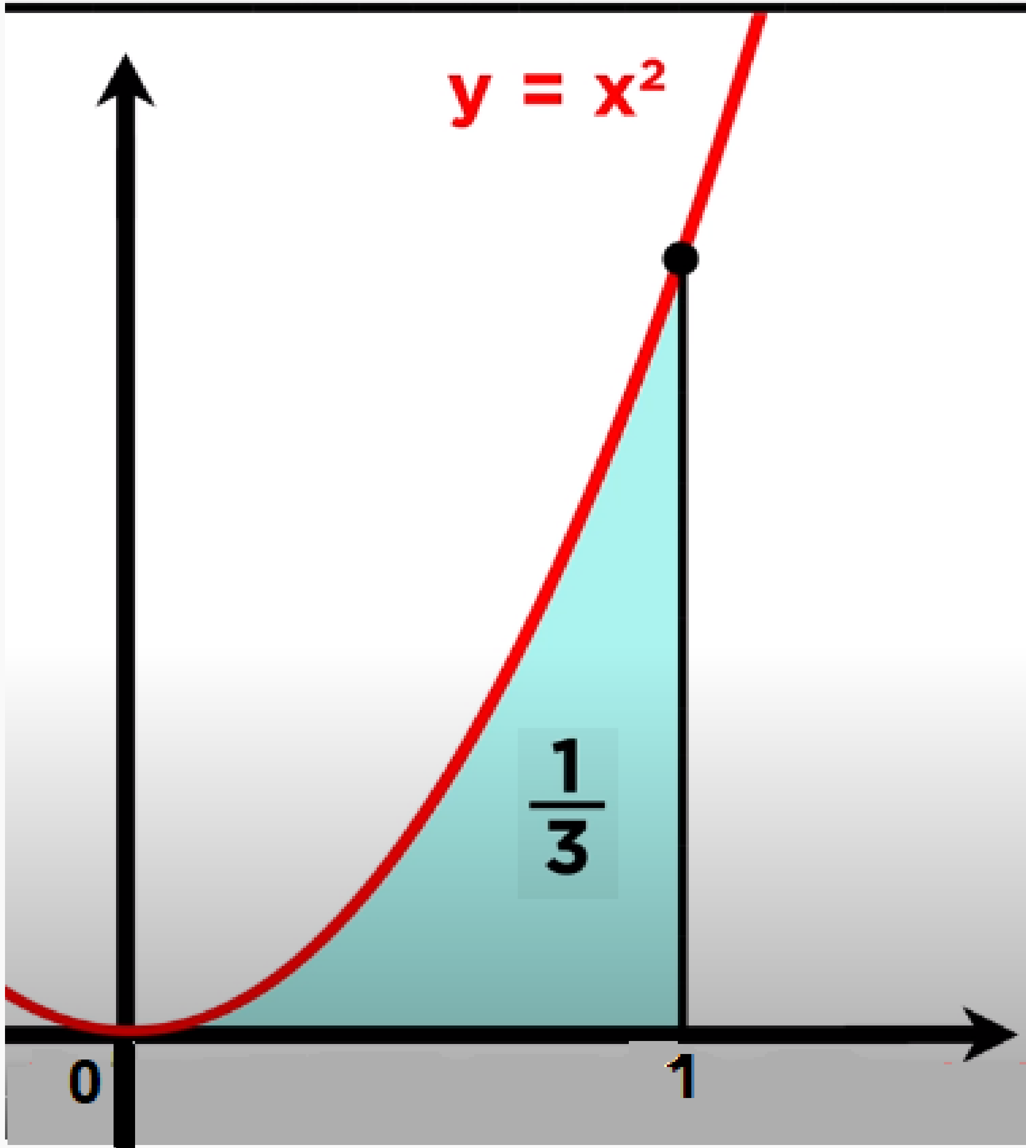
$$g(x) = \int_a^x f(t)\,dt$$

we can evaluate the **antiderivative** at these limits and subtract to get the specific **area**

$$g(x) = F(x) - F(a)$$

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Finding the Antiderivative of a Function



$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1$$

$$\frac{(1)^3}{3} - \frac{(0)^3}{3}$$

$$\frac{1}{3} - 0 = \frac{1}{3}$$

Ex 7.9, 4

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

Let $F(x) = \int \sin 2x \, dx$

$$= \frac{1}{2}(-\cos 2x)$$

$$= \frac{-1}{2}\cos 2x$$

$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= \frac{-1}{2}\cos\left(2 \times \frac{\pi}{4}\right) - \left(\frac{-1}{2}\cos(2 \times 0)\right)$$

$$= \frac{-1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0$$

$$= \frac{-1}{2} \times 0 + \frac{1}{2} \times 1 = \mathbf{\frac{1}{2}}$$

Ex 7.9, 17

$$\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2) \, dx$$

Let $F(x) = \int (2\sec^2 x + x^3 + 2) \, dx$

$$= 2\int \sec^2 x \, dx + \int x^3 \, dx + \int 2 \, dx$$

$$= 2\tan x + \frac{x^4}{4} + 2x$$

$$\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2) dx = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= \left[2\tan\frac{\pi}{4} + \frac{\left(\frac{\pi}{4}\right)^4}{4} + 2\frac{\pi}{4}\right] - \left[2\tan(0) + \frac{(0)^4}{4} + 2 \times 0\right]$$

$$= 2 \times 1 + \frac{\pi^4}{4^5} + \frac{\pi}{2} - 0$$

$$= \mathbf{2 + \frac{\pi^4}{1024} + \frac{\pi}{2}}$$

Ex 7.9, 19

$$\int_0^2 \frac{6x+3}{x^2+4}\,dx$$

Let $F(x) = \int \frac{6x+3}{x^2+4}\,dx$

$$= \int \frac{6x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx$$

$\downarrow$ I_1 $\qquad$ $\downarrow$ I_2

Solving I_1 : Put $x^2 + 4 = t$

$2x\ dx = dt$

$$\int \frac{6x}{x^2+4}\,dx = \int \frac{3}{t}\,dt = 3\log|t|$$

$$= 3\log|x^2+4|$$

Solving I_2 : $3\int \frac{1}{x^2+4}\,dx = 3\int \frac{1}{x^2+2^2}\,dx$

$$= \frac{3}{2}\tan^{-1}\frac{x}{2}$$

Therefore

$$F(x) = I_1 + I_2$$

$$F(x) = 3\log|x^2+4| + \frac{3}{2}\tan^{-1}\frac{x}{2}$$

Now,

$$\int_0^2 \frac{6x+3}{x^2+4}\,dx = F(2) - F(0)$$

$$= 3\log|2^2+4| + \frac{3}{2}\tan^{-1}\frac{2}{2} - 3\log|0+4| - \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)$$

$$= 3\log|4+4| + \frac{3}{2}\tan^{-1}1 - 3\log|4| - \frac{3}{2}\times 0$$

$$= 3\log|8| - 3\log|4| + \frac{3}{2}\,\frac{\pi}{4}$$

$$= 3(\log|8| - \log|4|) + \frac{3\pi}{8}$$

$$= \mathbf{3\log 2 + \frac{3\pi}{8}}$$ (As $\log A - \log B = \log\frac{A}{B}$)

Ex 7.9, 20

$\int_0^1 \left(x\, e^x + \sin \frac{\pi x}{4}\right)\, dx$

Let $F(x) = \int \left(xe^x + sin \frac{\pi x}{4}\right) dx$

$= \int xe^x\, dx + \int \sin\left(\frac{\pi x}{4}\right)\, dx$

$\downarrow$ I_1 $\quad$ $\downarrow$ I_2

Solving I_1 : $\int xe^x\, dx$

$= xe^x - \int (1. e^x dx) dx$

$= xe^x - \int e^x\, dx$

$= xe^x - e^x$

$= e^x(x-1)$

Solving by parts

$\int u\, v\, dx = u \int v\, dx - \int \left(u' \int v\, dx\right) dx$

Let $u = x$ and $v = e^x$

Solving I_2 : $\int \sin\left(\frac{\pi x}{4}\right)\, dx$

$= \frac{1}{\frac{\pi}{4}}\left(-\cos\left(\frac{\pi x}{4}\right)\right)$

$= \frac{-4}{\pi} \cos\left(\frac{\pi x}{4}\right)$

Therefore,

$F(x) = \int xe^x\, dx + \int sin \frac{\pi}{4} x\, dx$

$= e^x(x-1) - \frac{4}{\pi} \cos\left(\frac{\pi x}{4}\right)$

Now,

$\int_0^1 \left(xe^x + sin \frac{\pi x}{4}\right) dx = F(1) - F(0)$

$= e \times 0 - \frac{4}{\pi} cos \frac{\pi}{4} - 1(-1) + \frac{4}{\pi} \cos 0$

$= \frac{-4}{\pi} cos \frac{\pi}{4} + 1 + \frac{4}{\pi}$

$= \frac{-4}{\pi} \frac{1}{\sqrt{2}} + 1 + \frac{4}{\pi}$

$= \frac{-2\sqrt{2}}{\pi} + 1 + \frac{4}{\pi}$

$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION METHOD

Ex 7.10, 2

Evaluate the integrals using substitution

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\ \phi}\ \cos^5 \phi\, d\phi$$

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{sin\phi}\, \cos^5 \phi\ d\phi$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{sin\phi}\, \cos^4 \phi\ cos\phi\ d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, (1-\sin^2 \phi)^2 cos\phi\ d\phi$$

Put $t = \sin \phi$

$dt = \cos \phi\ d\phi$

when ϕ varies form 0 to $\frac{\pi}{2}$,

t varies form 0 to 1

ϕ	$t = sin\ \phi$
0	$t = \sin 0 = 0$
$\frac{\pi}{2}$	$t = \sin\frac{\pi}{2} = 1$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{sin\phi}\, (1-\sin^2 \phi)^2 cos\phi\ d\phi$$

$$= \int_0^1 \sqrt{t}\, (1-t^2)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}} (1-2t^2+t^4) dt$$

$$= \int_0^1 t^{\frac{1}{2}}\, dt - 2\int t^{\frac{3}{2}}\ dt + \int t^{\frac{9}{2}}\ dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - 2\left[\frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1 + \left[\frac{t^{\frac{11}{2}}}{\frac{11}{2}}\right]_0^1$$

$$= \frac{2}{3}\left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) - 2 \times \frac{2}{7}\left(1^{\frac{7}{2}} - 0^{\frac{7}{2}}\right) + \frac{2}{11}\left[1^{\frac{11}{2}} - 0^{\frac{11}{2}}\right]$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \quad = \frac{64}{231}$$

Ex 7.10, 3

Evaluate the integrals using substitution

$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Put x = tan ϕ

Differentiating w.r.t.ϕ

$dx = sec^2\phi \, d\phi$

x	$\phi = tan^{-1} x$
0	$\phi = tan^{-1}0 = 0$
1	$\phi = tan^{-1}1 = \frac{\pi}{4}$

when x varies from 0 to 1,

ϕ varies from 0 to $\frac{\pi}{4}$

$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2tan\phi}{1+tan^2\phi}\right) sec^2\phi \, d\phi$

(Using $\frac{2tan\phi}{1+tan^2\phi} = \sin 2\phi$)

$I = \int_0^{\frac{\pi}{4}} \sin^{-1}(sin2\phi) \, sec^2\phi \, d\phi$

$I = \int_0^{\frac{\pi}{4}} 2\phi \, sec^2\phi \, d\phi$

$I = 2\int_0^{\frac{\pi}{4}} \phi \, sec^2\phi \, d\phi$

Algebraic **Trigonometric**

Using by parts

$\int_a^b u \, v \, dx = u\int_a^b v \, dx - \int_a^b u' \int_a^b v \, dx) \, dx$

Putting $u = \phi$, $v = sec^2\phi$

$I = 2 \times [\phi \tan\phi - \int 1 \times \tan\phi \, d\phi]_0^{\frac{\pi}{4}}$

$= 2 \times [\phi \tan\phi - log|\sec\phi|]_0^{\frac{\pi}{4}}$

$= 2\left[\frac{\pi}{4} tan\frac{\pi}{4} - log\left|sec\left(\frac{\pi}{4}\right)\right| - (0\tan(0) - log|\sec(0)|)\right]$

$= 2\left[\frac{\pi}{4} \times 1 - log|\sqrt{2}| - 0 + log\,|1|\right]$

$= 2\left(\frac{\pi}{4} - log\sqrt{2} - 0 + 0\right)$ $(\because \log 1 = 0)$

$= \frac{\pi}{2} - 2\,log\,\sqrt{2}$

$= \frac{\pi}{2} - \log\left(\sqrt{2}\right)^2$ $(\because a\,logx = logx^a)$

$= \frac{\pi}{2} - \boldsymbol{log\ 2}$

Ex7.10, 7

Evaluate the integrals using substitution

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$$

we can write

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \int_{-1}^{1} \frac{dx}{(x + 2x + 1) + 4}$$

$$= \int_{-1}^{1} \frac{dx}{(x + 1)^2 + 2^2}$$

Putting $x + 1 = t$

$dx = dt$

when x varies from -1 to 1

then t varies from 0 to 2

x	$t = x + 1$
-1	$t = -1 + 1 = 0$
1	$t = 1 + 1 = 2$

Therefore,

$$\int_{-1}^{1} \frac{dx}{(x+1)^2 + 2^2} = \int_{0}^{2} \frac{dt}{t^2 + 2^2}$$

We know that

$$\int_{q}^{p} \frac{dx}{t^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{t}{a}\right]_q^p$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}\frac{2}{2} - \frac{1}{2}\tan^{-1}\frac{0}{2}$$

$$= \frac{1}{2}\tan^{-1} 1 - \frac{1}{2}\tan^{-1} 0$$

$$= \frac{1}{2} \times \frac{\pi}{4} - 0$$

$$= \frac{\pi}{8}$$

Ex 7.10, 9

The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4}\, dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

Taking common x^3 from numerator

$$= \int_{\frac{1}{3}}^{1} \frac{(x^3)^{\frac{1}{3}}\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^4}\, dx$$

$$= \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^4}\, dx$$

$$= \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^3}\, dx$$

Let $t = \frac{1}{x^2} - 1$

$dt = \frac{-2}{x^3}\, dx$

when x varies from $\frac{1}{3}$ to 1,

t varies form 0 to 8

x	$t = \frac{1}{x^2} - 1$
$\frac{1}{3}$	$t = \frac{1}{x^2} - 1 = 3^2 - 1 = 8$
1	$t = \frac{1}{x^2} - 1 = \frac{1}{1} = 0$

$$\int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^3}\, dx = \frac{1}{2}\int_{8}^{0} t^{\frac{1}{3}} dt$$

$$= \frac{-1}{2}\left[\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_8^0$$

$$= \frac{-1}{2}\left[\frac{3t^{\frac{4}{3}}}{4}\right]_8^0$$

Putting limits

$$= \frac{-1}{2}\left(0 - \frac{3(8)^{\frac{4}{3}}}{4}\right)$$

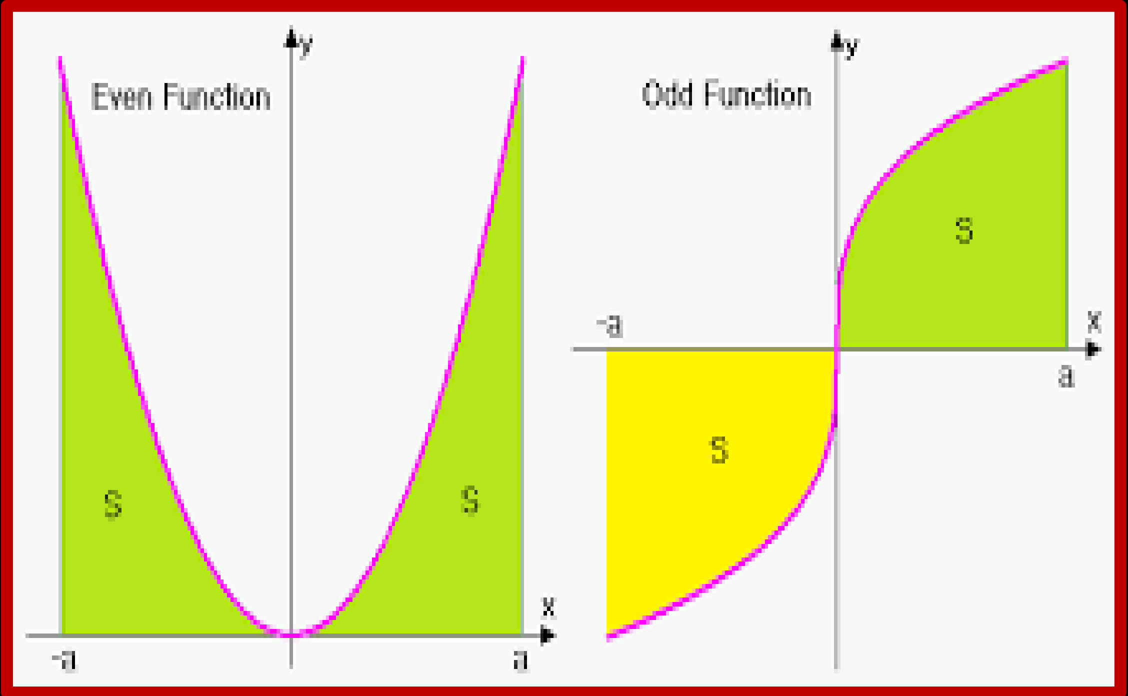
$$= \frac{1}{2}\left(\frac{3}{4}\right)(8)^{\frac{4}{3}}$$

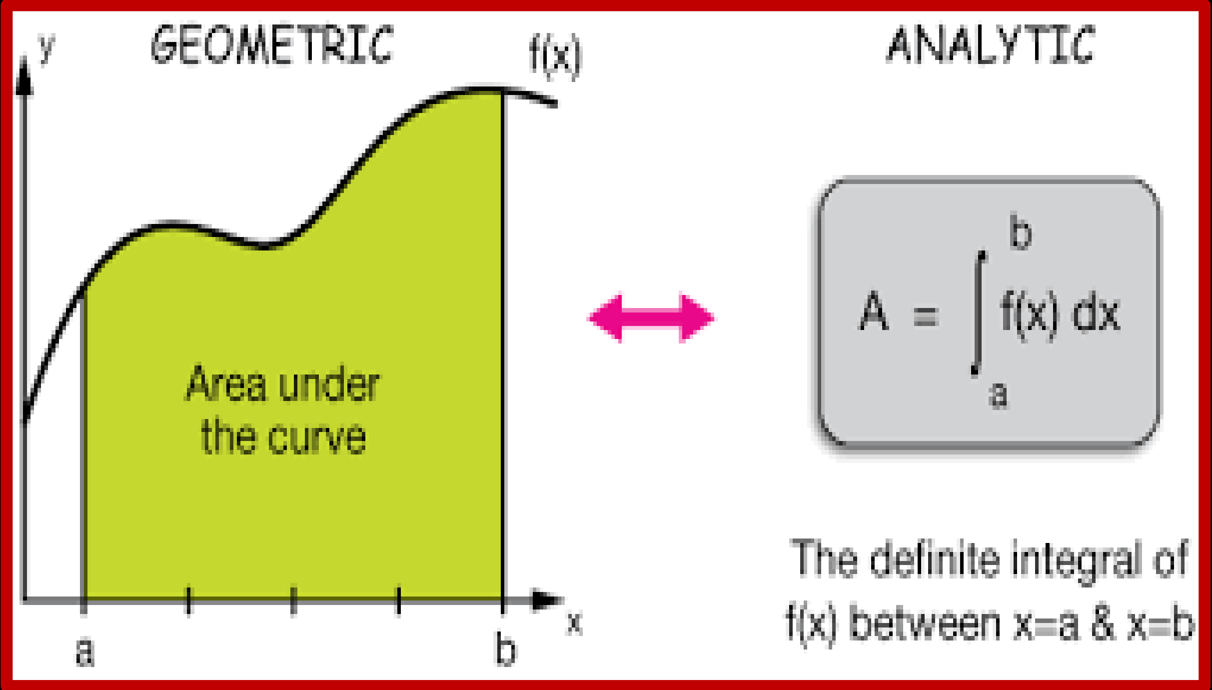
$$= \frac{1}{2}\left(\frac{3}{4}\right)(2^3)^{\frac{4}{3}} = \frac{1}{2}\left(\frac{3}{4}\right)(2^4) = 6$$

So, **(A) is the correct answer.**

HOME ASSIGNMENT…..

EXERCISE 7.9 – Q.NO. 3, 5, 7, 8,12, 14, 16, 18
EXERCISE 7.10 – Q.NO. 1, 4, 5, 8.





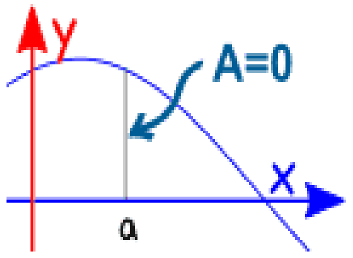
INTEGRALS

MODULE – 9

PROPERTIES OF DEFINITE INTEGRALS :

I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

In particular, $\int_a^a f(x)\,dx = 0$



Proof : Put $x = t \implies dx = dt$

when $x = a,\ t = a$ and when $x = b,\ t = b.$

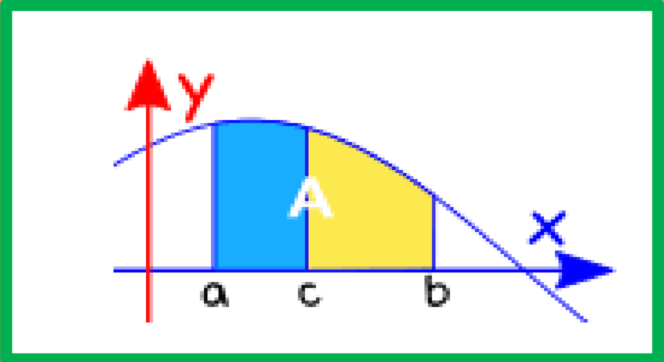
$\therefore \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof : $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) = -[F(a) - F(b)] = -[F(x)]_b^a$

$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



Proof : $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ ...... (1)

$\int_a^c f(x)dx = [F(x)]_a^c = F(c) - F(a)$ ...... (2)

$\int_c^b f(x)dx = [F(x)]_c^b = F(b) - F(c)$ ...... (3)

Adding (2) and (3) ,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= [F(x)]_a^b = \int_a^b f(x)dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex 7.11, 5

By using the properties of definite integrals, evaluate the integrals :

$\int_{-5}^{5} |x+2|\, dx$

$$|x+2| = \begin{cases} (x+2) & if\ x+2 \geq 0 \\ -(x+2) & if\ x+2 < 0 \end{cases} = \begin{cases} (x+2)\ if\ x \geq -2 \\ -(x+2)\ if\ x < -2 \end{cases}$$

> *Using the Property*
>
> $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$\therefore \int_{-5}^{5}|x+2|dx = \int_{-5}^{-2}|x+2|dx + \int_{-2}^{5}|x+2|dx$$

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx$$

$$= -\int_{-5}^{-2} xdx - \int_{-5}^{-2} 2dx + \int_{-2}^{5} xdx + \int_{-2}^{5} 2dx$$

$$= -\left[\frac{x^2}{2}\right]_{-5}^{-2} - 2[x]_{-5}^{-2} + \left[\frac{x^2}{2}\right]_{-2}^{5} + 2[x]_{-2}^{5}$$

$$= -\left(\frac{(-2)^2-(-5)^2}{2}\right) - 2[-2-(-5)] + \left[\frac{(5)^2-(-2)^2}{2}\right] + 2[5-(-2)]$$

$$= -\left(\frac{4-25}{2}\right) - 2[-2+5] + \left[\frac{25-4}{2}\right] + 2[5+2]$$

$$= -\left(\frac{-21}{2}\right) - 2[3] + \frac{21}{2} + 2[7]$$

$$= \frac{21}{2} + \frac{21}{2} - 6 + 14 = 21 + 8 = 29$$

IV. $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof : Let $t = a + b - x \implies dt = -dx$

when $x = a, t = b$ and when $x = b, t = a$

$$\int_a^b f(x)dx = \int_b^a f(a+b-t)(-dt)$$

$$= -\int_b^a f(a+b-t)dt$$

$$= \int_a^b f(a+b-t)dt$$

$$= \int_a^b f(a+b-x)dx$$

Example 35

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} . dx$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\frac{\sin x}{\cos x}}} . dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}}} dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx \qquad ...(1)$$

Using the ***Property***

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$\therefore I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}} dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

Using :-

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\& \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \qquad ...(2)$$

Adding (1) and (2)

$$I + I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} . dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1.\ dx$$

$$2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$I = \frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{1}{2}\left[\frac{2\pi-\pi}{6}\right] = \frac{\pi}{12}$$

V. $$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Proof : Let $t = a - x \implies dt = -dx$

when $x = 0, t = a$ and $x = a, t = 0.$

$$\int_0^a f(x)dx = -\int_a^0 f(t)\,dt$$

$$= \int_0^a f(t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Ex 7.11, 2

By using the properties of definite integrals, evaluate the integrals :

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ...(1)

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

As $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ ...(2)

Adding (1) and (2)

$$I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right] dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$I = \frac{1}{2}[x]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$\therefore$ $\boldsymbol{I = \frac{\pi}{4}}$

Ex 7.11,7

By using the properties of definite integrals,

evaluate the integrals : $\int_0^1 x(1-x)^n\,dx$

Let $I = \int_0^1 x(1-x)^n\,dx$

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore \quad I = \int_0^1 (1-x)[1-(1-x)]^n\,dx$

$I = \int_0^1 (1-x)[1-1+x]^n\,dx$

$I = \int_0^1 (1-x)\,x^n\,dx$

$I = \int_0^1 (x^n - x^{n+1})\,dx$

$I = \int_0^1 x^n\,dx - \int_0^1 x^{n+1}\,dx$

$$I = \left[\frac{x^{n+1}}{n+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1$$

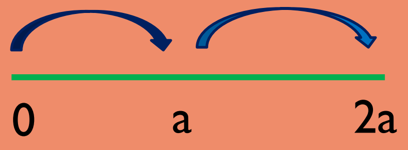
$$I = \left[\frac{(1)^{n+1}}{n+1} - \frac{(0)^{n+1}}{n+1}\right] - \left[\frac{(1)^{n+2}}{n+2} - \frac{(0)^{n+2}}{n+2}\right]$$

$$I = \frac{1}{n+1} - \frac{1}{n+2}$$

$$I = \frac{n+2-(n+1)}{(n+1)(n+2)}$$

$$\mathbf{I = \frac{1}{(n+1)(n+2)}}$$

VI. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$



Proof : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

Let $t = 2a - x \Rightarrow dt = -dx$

when $x = a, t = a$ and when $x = 2a, t = 0$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)\,(-dt)$$

$$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Ex 7.11, 14

By using the properties of definite integrals, evaluate the integrals :

$\int_0^{2\pi} \cos^5 x \ dx$

Using property: $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$

$\int_0^{2\pi} \cos^5 x \ dx$

$= \int_0^{\pi} \cos^5 x \ dx + \int_0^{\pi} \cos^5(2\pi - x) \ dx$

$= \int_0^{\pi} cos^5 x \, dx + \int_0^{\pi} cos^5 \ x$ (As $\cos(2\pi - \theta) = \cos\theta$)

$= 2\int_0^{\pi} cos^5 x \, dx$

Using property: $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$

$= 2\left(\int_0^{\frac{\pi}{2}} cos^5 x \, dx + \int_0^{\frac{\pi}{2}} \cos(\pi - x) \ dx\right)$

$(\cos(\pi - \theta) = -\cos\theta)$

$= 2\left(\int_0^{\frac{\pi}{2}} cos^5 x \, dx + \int_0^{\frac{\pi}{2}} (-\cos x)^5 \, dx\right)$

$= 2\left(\int_0^{\frac{\pi}{2}} cos^5 x \, dx - \int_0^{\frac{\pi}{2}} cos^5 x \, dx\right)$

$= 2 \times 0$

$= 0$

VII. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx, & if\ f(2a-x) = f(x) \quad and \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$$

Proof : $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$ …….(1)

Now if $f(2a-x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

and if $f(2a-x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx - \int_0^{a} f(x)\,dx = 0$$

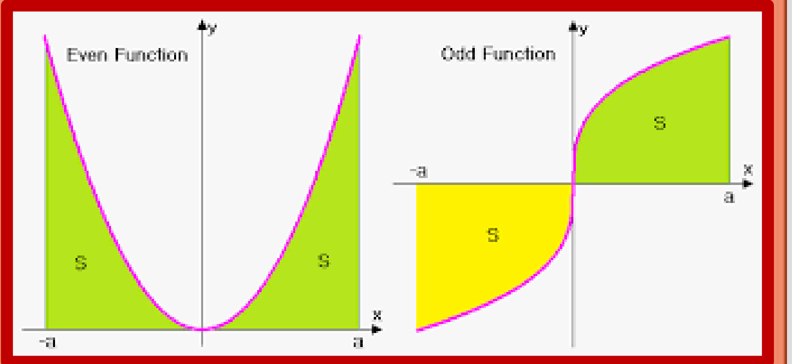
VIII. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f \text{ is an even function, i.e, } f(-x) = f(x) \\ 0, & \text{if } f \text{ is an odd function, i.e, } f(-x) = -f(x) \end{cases}$

Proof : $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

Let $t = -x \Rightarrow dt = -dx$ when $x = -a,\ t = a$ & $x = a,\ t = 0.$

$$= -\int_a^0 f(-t)\,dt + \int_0^a f(x)dx$$

$$= \int_0^a f(-t)\,dt + \int_0^a f(x)dx$$



$\therefore \int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx \ldots\ldots.(1)$

(i) Now, if f is an even function, then $\boldsymbol{f(-x) = f(x)}$ and so (1) becomes

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)\,dx$$

(ii) If f is an odd function, then $\boldsymbol{f(-x) = -f(x)}$ and so (1) becomes

$$\int_{-a}^{a} f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$$

Ex 7.11, 11

By using the properties of definite integrals,

evaluate the integrals : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \; dx$

This is of form $\int_{-a}^{a} f(x)dx$ where

$f(x) = \sin^2 x$

$f(-x) = \sin^2(-x) = (-sinx)^2 = \sin^2 x$

$\therefore \quad f(x) = f(-x)$

Using the Property

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \;\; if \; f(-x) = f(x)$

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx = 2\int_{0}^{\frac{\pi}{2}} \boldsymbol{\sin^2 x}\, dx$

$\because \cos 2x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x = 1 - \cos 2x$

$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$

$= 2\int_{0}^{\frac{\pi}{2}} \left[\frac{\mathbf{1 - \cos 2x}}{\mathbf{2}}\right] dx$

$= \int_{0}^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$

$= \left[x - \frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$

$= \left[\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2}\right] - \left[0 - \frac{\sin 2(0)}{2}\right]$

$= \frac{\pi}{2} - \frac{\sin \pi}{2} - 0$

$= \frac{\pi}{2} - 0 + 0 \quad = \frac{\pi}{2}$

Example 36

Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \; dx$

Let $I_1 = \int_0^{\frac{\pi}{2}} log(sinx)\, dx$ ...(1)

Using Property

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore \quad I_1 = \int_0^{\frac{\pi}{2}} sin\left(\frac{\pi}{2} - x\right) dx$

$I_1 = \int_0^{\frac{\pi}{2}} log(\cos x) dx$ ...(2)

Adding (1) and (2)

$$I_1 + I_1 = \int_0^{\frac{\pi}{2}} log(\sin x)dx + \int_0^{\frac{\pi}{2}} log(\cos x)dx$$

$2I_1 = \int_0^{\frac{\pi}{2}} \log[\sin x \cos x]\, dx$ (Using $\log a + \log b = \log(a.b)$)

$$2I_1 = \int_0^{\frac{\pi}{2}} \log\left[\frac{2\sin x \cos x}{2}\right] dx$$

$2I_1 = \int_0^{\frac{\pi}{2}} [\log[\sin 2x] - \log 2] dx$ (Using $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$)

$$2I_1 = \int_0^{\frac{\pi}{2}} \log[\sin 2x]dx - \int_0^{\frac{\pi}{2}} \log 2\, dx \qquad ...(3)$$

↓

I_2

Solving $I_2 = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx$

Let $2x = t$

Differentiating both sides w.r.t. x

$2\, dx = dt$

x	$t = 2x$
0	$t = 2(0) = 0$
$\frac{\pi}{2}$	$t = 2\left(\frac{\pi}{2}\right) = \pi$

∴ Putting the values of t and dt and changing the limits,

$$I_2 = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt$$

Using the Property

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx\, , if\ f(2a-x) = f(x)$$

Here, $f(t) = \log sint$

$$f(2a-t) = f(2\pi - t) = \log sin(2\pi - t) = \log \sin t$$

Since $f(t) = f(2a - t)$

$$\therefore I_2 = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log \sin t. dt \quad = \int_0^{\frac{\pi}{2}} \log \sin t. dt$$

Now, Using the Property

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$I_2 = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

Putting the value of I_2 in equation (3), we get

$$2I_1 = \int_0^{\frac{\pi}{2}} \mathbf{\log[\mathit{sin}\,2\mathit{x}]\,\mathit{dx}} - \int_0^{\frac{\pi}{2}} \log(2)\, dx$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \mathbf{\log(\mathit{sin}\,\mathit{x})\,\mathit{dx}} - \log(2) \int_0^{\frac{\pi}{2}} 1. dx$$

$$2I_1 = \mathbf{I_1} - \log(2)\, [x]_0^{\frac{\pi}{2}}$$

$$2I_1 - I_1 = -\log 2 \left[\frac{\pi}{2} - 0\right]$$

$$I_1 = -\log 2 \left[\frac{\pi}{2}\right] \qquad \therefore \qquad \mathbf{I_1 = \frac{-\pi}{2} \log 2}$$

HOME ASSIGNMENT

EXERCISE 7.11

Q.NO. 3, 4, 6, 8, 9, 10, 13, 16, 18.